

Causal Bulk Viscous Dissipative Isotropic Cosmologies with Variable Gravitational and Cosmological Constants

M. K. Mak*

Department of Physics, The Hong Kong University of Science and Technology, Clear Water Bay, Hong Kong, P. R. China.

J. A. Belinchón†

Grupo Inter-Universitario de Análisis Dimensional, Dept. Fisica ETS Arquitectura UPM, Madrid, Spain.

T. Harko‡

Department of Physics, The University of Hong Kong, Pokfulam Road, Hong Kong, P. R. China.

We consider the evolution of a flat Friedmann-Robertson-Walker Universe, filled with a causal bulk viscous cosmological fluid, in the presence of variable gravitational and cosmological constants. The basic equation for the Hubble parameter, generalizing the evolution equation in the case of constant gravitational coupling and cosmological term is derived, under the supplementary assumption that the total energy of the Universe is conserved. By assuming that the cosmological constant is proportional to the square of the Hubble parameter and a power law dependence of the bulk viscosity coefficient, temperature and relaxation time on the energy density of the cosmological fluid, two classes of exact solutions of the field equations are obtained. In the first class of solutions the Universe ends in an inflationary era, while in the second class of solutions the expansion of the Universe is noninflationary for all times. In both models the cosmological "constant" is a decreasing function of time, while the gravitational "constant" increases in the early period of evolution of the Universe, tending in the large time limit to a constant value.

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I. INTRODUCTION

Recent observations of type Ia supernovae with redshift up to about $z \lesssim 1$ provided evidence that we may live in a low mass-density Universe, with the contribution of the non-relativistic matter (baryonic plus dark) to the total energy density of the Universe of the order of $\Omega_m \sim 0.3$ [1]- [3]. The value of Ω_m is significantly less than unity [4] and consequently either the Universe is open or there is some additional energy density ρ sufficient to reach the value $\Omega_{total} = 1$, predicted by inflationary theory. Observations also show that the deceleration parameter of the Universe q is in the range $-1 \leq q < 0$, and the present-day Universe undergoes an accelerated expansionary evolution.

Several physical models have been proposed to give a consistent physical interpretation to these observational facts. One candidate, and maybe the most convincing one for the missing energy is vacuum energy density or cosmological constant Λ [5].

Since the pioneering work of Dirac [6], who proposed, motivated by the occurrence of large numbers in Universe, a theory with a time variable gravitational coupling constant G , cosmological models with variable G and nonvanishing and variable cosmological term have been intensively investigated in the physical literature [7]- [27]. In the isotropic cosmological model of Chen and Wu [14] it is supposed, in the spirit of quantum cosmology, that the effective cosmological constant Λ varies as a^{-2} (with a the scale function). In the cosmological model of Lima and Maia [26] the cosmological constant $\Lambda = \Lambda(H) = 3\beta H^2 + 3(1-\beta)H^3/H_I$ is a complicated function of the Hubble parameter H , a constant β and an arbitrary time scale H_I^{-1} , leading to a cosmic history beginning from an instability of the de Sitter space-time. The cosmological implications of a time dependence of the cosmological of the form $\Lambda \sim t^{-2}$ have been considered by Berman [12]. Waga [27] investigated flat cosmological models with the cosmological term varying as $\Lambda = \alpha/a^2 + \beta H^2 + \gamma$, with α , β and γ constants. In this model exact expressions for observable quantities can be obtained. Nucleosynthesis in decaying-vacuum cosmological models based on the Chen-Wu ansatz [14] has been investigated by Abdel-Rahman [8]. The consistency with the observed helium abundance and baryon asymmetry allows a maximum vacuum energy close to the radiation energy today. Anisotropic Bianchi type I cosmological models with variable G and Λ have been analyzed by Beesham [11] and it was shown that in this case there are no

*E-mail:mkmak@vtc.edu.hk

†E-mail: abelcal@ciccp.es

‡E-mail: tcharko@hkusua.hku.hk

classical inflationary solutions of pure exponential form. Cosmological models with the gravitational and cosmological constants generalised as coupling scalars and with $G \sim a^n$ have been discussed by Sistero [25]. Generalized field equations with time dependent G and Λ have been proposed in [19] and [20] in an attempt to reconcile the large number hypothesis with Einstein's theory of gravitation. Limits on the variability of G using binary-pulsar data have been obtained by Damour, Gibbons and Taylor [15]. A detailed analysis of Friedmann-Robertson-Walker (FRW) Universes in a wide range of scalar-tensor theories of gravity has been performed by Barrow and Parsons [10].

Dissipative effects, including both bulk and shear viscosity, are supposed to play a very important role in the early evolution of the Universe. The first attempts at creating a theory of relativistic fluids were those of Eckart [28] and Landau and Lifshitz [29]. These theories are now known to be pathological in several respects. Regardless of the choice of equation of state, all equilibrium states in these theories are unstable and in addition signals may be propagated through the fluid at velocities exceeding the speed of light. These problems arise due to the first order nature of the theory, that is, it considers only first-order deviations from the equilibrium leading to parabolic differential equations, hence to infinite speeds of propagation for heat flow and viscosity, in contradiction with the principle of causality. Conventional theory is thus applicable only to phenomena which are quasi-stationary, i.e. slowly varying on space and time scales characterized by mean free path and mean collision time.

A relativistic second-order theory was found by Israel [30] and developed by Israel and Stewart [31], Hiscock and Lindblom [32] and Hiscock and Salmonson [33] into what is called "transient" or "extended" irreversible thermodynamics. In this model deviations from equilibrium (bulk stress, heat flow and shear stress) are treated as independent dynamical variables, leading to a total of 14 dynamical fluid variables to be determined. For general reviews on causal thermodynamics and its role in relativity see [34] and [35].

Causal bulk viscous thermodynamics has been extensively used for describing the dynamics and evolution of the early Universe or in an astrophysical context. But due to the complicated character of the evolution equations, very few exact cosmological solutions of the gravitational field equations are known in the framework of the full causal theory. For a homogeneous Universe filled with a full causal viscous fluid source obeying the relation $\xi \sim \rho^{1/2}$, with ρ the energy density of the cosmological fluid, exact general solutions of the field equations have been obtained in [36]- [44]. It has also been proposed that causal bulk viscous thermodynamics can model on a phenomenological level matter creation in the early Universe [39], [41]. Exact causal viscous cosmologies with $\xi \sim \rho^s, s \neq 1/2$ have been considered in [40].

Because of technical reasons, most investigations of dissipative causal cosmologies have assumed FRW symmetry (i.e. homogeneity and isotropy) or small perturbations around it [45]. The Einstein field equations for homogeneous models with dissipative fluids can be decoupled and therefore are reduced to an autonomous system of first order ordinary differential equations, which can be analyzed qualitatively [46], [47], [48].

Recent developments in particle physics and cosmology have shown that the cosmological constant Λ ought to be treated as a dynamical quantity rather than a simple constant. The dynamics of the scale factor in FRW type models with a variable Λ term has been recently revisited for the perfect fluid case by Overduin and Cooperstock [49].

The effects of dissipation as expressed in the form of a non-vanishing bulk viscosity coefficient in the stress-energy tensor of the matter in cosmological models with variable Λ and G have been considered by several authors [50]- [54]. The role of a transient bulk viscosity in a FRW space-time with decaying vacuum has been discussed in [50]. Models with causal bulk viscous cosmological fluid have been considered recently by Arbab and Beesham [54]. They obtained both power-law and inflationary solutions, with the gravitational constant an increasing function of time.

It is the purpose of the present paper to consider the evolution of causal bulk viscous dissipative cosmological models in the presence of variable gravitational and cosmological constants. By assuming that Λ is a quadratic function of the Hubble parameter H and that the total energy of the Universe is conserved, the gravitational field equations can be exactly integrated and two classes of solutions are obtained. The first class describes a barotropic viscous fluid with arbitrary γ , while the second solution corresponds to a stiff cosmological fluid.

The present paper is organised as follows. The basic equations of the model are written down in Section II. In Section III we present the first class of exact solutions. The case of the stiff bulk viscous cosmological fluid is considered in Section IV. In Section V we discuss and conclude our results.

II. GEOMETRY, FIELD EQUATIONS AND CONSEQUENCES

We consider that the geometry of the Universe filled with a bulk viscous cosmological fluid can be described by a spatially flat FRW type metric given by

$$ds^2 = dt^2 - a^2(t) [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)]. \quad (1)$$

The Einstein gravitational field equations with variable G and Λ are:

$$R_{ik} - \frac{1}{2}g_{ik}R = 8\pi G(t)T_{ik} + \Lambda(t)g_{ik}. \quad (2)$$

In the following we consider a system of units so that $c = 1$.

The bulk viscous effects can be generally described by means of an effective pressure Π , formally included in the effective thermodynamic pressure p_{eff} [34]. Then in the comoving frame the energy momentum tensor has the components $T_0^0 = \rho, T_1^1 = T_2^2 = T_3^3 = -p_{eff}$. For the line element (1) the Einstein field equations give:

$$3\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G(t)\rho + \Lambda(t), \quad (3)$$

$$3\frac{\ddot{a}}{a} = -4\pi G(t)(3p_{eff} + \rho) + \Lambda(t), \quad (4)$$

where a dot denotes the derivative with respect to the time t .

Taking the covariant derivative of the Einstein field equations with respect to the FRW metric (1) we obtain the general conservation law in the presence of variable gravitational and cosmological constants:

$$3H(p_{eff} + \rho) = -\left(\frac{\dot{G}(t)}{G(t)}\rho + \dot{\rho} + \frac{\dot{\Lambda}(t)}{8\pi G(t)}\right), \quad (5)$$

where we have also introduced the Hubble parameter $H = \dot{a}/a$.

Assuming that the total matter content of the Universe is conserved, $T_{i;j}^{(mat)j} = 0$, the energy density of the matter obeys the usual conservation law:

$$\dot{\rho} + 3H(p_{eff} + \rho) = 0. \quad (6)$$

In the presence of the bulk viscous stress Π , the effective thermodynamic pressure term becomes $p_{eff} = p + \Pi$, where p is the thermodynamic pressure of the cosmological fluid. Then the energy conservation equation (5) can be split into two independent equations:

$$\dot{\rho} + 3H(p + \rho) = -3\Pi H, \quad (7)$$

$$8\pi\dot{G}\rho + \dot{\Lambda} = 0. \quad (8)$$

The causal evolution equation for the bulk viscous pressure is given by [34]

$$\tau\dot{\Pi} + \Pi = -3\xi H - \frac{1}{2}\tau\Pi\left(3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T}\right), \quad (9)$$

where T is the temperature, ξ the bulk viscosity coefficient and τ the relaxation time.

The growth of the total comoving entropy $\Sigma(t)$ over a proper time interval (t_0, t) is

$$\Sigma(t) - \Sigma(t_0) = -\frac{3}{k_B} \int_{t_0}^t \frac{\Pi a^3 H}{T} dt, \quad (10)$$

where k_B is the Boltzmann constant.

An important observational quantity is the deceleration parameter $q = \frac{d}{dt}\left(\frac{1}{H}\right) - 1$. The sign of the deceleration parameter indicates whether the model inflates or not. The positive sign of q corresponds to “standard” decelerating models whereas the negative sign indicates inflation. The deceleration parameter can be expressed as a function of the thermodynamic, gravitational and cosmological quantities in the form

$$q = \frac{4\pi G(t)(\rho + 3p + 3\Pi) - \Lambda(t)}{8\pi G(t)\rho + \Lambda(t)}. \quad (11)$$

Curvature is described by the tensor field R_{ijk}^l . It is well-known that if one uses singular behavior of the components of the tensor or its derivatives as a criterion for singularities, one gets into trouble since the singular behavior of components could be due to singular behavior of the coordinates or tetrad basis rather than the curvature itself. To

avoid this problem, one should examine the scalars formed out of curvature. The invariants $R_{ij}R^{ij}$ and $R_{ijkl}R^{ijkl}$ (the Kretschmann scalar) are very useful for the study of the singular behavior of the flat FRW metric:

$$R_{ij}R^{ij} = 12 \left(3H^4 + 3\dot{H}H^2 + \dot{H}^2 \right) \quad (12)$$

$$R_{ijkl}R^{ijkl} = 12 \left(2H^4 + 2\dot{H}H^2 + \dot{H}^2 \right) \quad (13)$$

In order to close the system of equations (3) and (7)-(9) we have first to give the equation of state for p and specify T , τ and ξ . We shall assume the following laws, [34]:

$$p = (\gamma - 1)\rho, \xi = \alpha\rho^s, T = T_0\rho^{\frac{\gamma-1}{\gamma}}, \tau = \frac{\xi}{\rho} = \alpha\rho^{s-1}, \quad (14)$$

where $1 \leq \gamma \leq 2$, $\alpha \geq 0$, $T_0 \geq 0$ and $s \geq 0$ are constants. Equations (14) are standard in the study of bulk viscous cosmological models, whereas the equation $\tau = \xi/\rho$ is a simple procedure to ensure that the speed of viscous pulses does not exceed the speed of light.

With these choices the general solution of the gravitational field equations with variable gravitational and cosmological constants still depends on the functional form of G and Λ . In the present paper we shall fix the mathematical form of the cosmological constant assuming that it is a function of the Hubble parameter only and its time dependence is:

$$\Lambda = 3\beta H^2. \quad (15)$$

This ansatz, initially proposed in [13] on dimensional ground has been widely used to study decaying vacuum cosmological models [27], [55]- [57].

From Eqs. (3), (8) and (15) it follows that in this model the gravitational constant G and the energy density ρ are given by:

$$G = bH^{-n\beta}, \rho = \rho_0 H^n, \quad (16)$$

where $\rho_0 = \frac{3}{4\pi b n} \geq 0$ and $b \geq 0$ are constants and $n = 2/(1-\beta) \geq 0$. For physically realistic cosmological models the condition $0 < \beta < 1$ must hold, in order to assure a time decreasing energy density of the Universe. One must also assume that H is a decreasing function of the cosmological time.

With the use of the barotropic equation of state $p = (\gamma - 1)\rho$ and Eq. (4), we obtain

$$\dot{H} + \frac{3\gamma}{n}H^2 + 4\pi G\Pi = 0. \quad (17)$$

In view of Eq. (16), Eq. (17) becomes

$$\dot{H} + \frac{3\gamma}{n}H^2 + 4\pi b H^{-n\beta}\Pi = 0. \quad (18)$$

With the use of Eqs. (14) and (18), the causal evolution equation for the bulk viscosity (9) leads to the following equation for the Hubble function H :

$$\ddot{H} + \left[3H + \alpha_0 H^{n(1-s)} \right] \dot{H} + \left(\frac{n-4\gamma}{2\gamma} \right) H^{-1}\dot{H}^2 + \frac{9}{n} \left(\frac{\gamma}{2} - 1 \right) H^3 + \frac{3\gamma\alpha_0}{n} H^{2+n(1-s)} = 0, \quad (19)$$

where we have denoted $\alpha_0 = (\rho_0)^{1-s}/\alpha$. Eq. (19) generalizes the standard evolution equation for causal bulk viscous models to the case of variable cosmological and gravitational constants and reduces to it in the case $G = \text{const.}$, $\Lambda = \text{const.}$

By introducing the transformations

$$H = y^{\frac{2\gamma}{n}}, \eta = \int y^{\frac{2\gamma}{n}} dt, \quad (20)$$

Eq. (19) reduces to

$$\frac{d^2y}{d\eta^2} + (3 + \alpha_0 y^m) \frac{dy}{d\eta} + \frac{9}{2\gamma} \left(\frac{\gamma}{2} - 1 \right) y + \frac{3\alpha_0}{2} y^{m+1} = 0, \quad (21)$$

where we have denoted $m = \frac{2\gamma[n(1-s)-1]}{n}$.

III. THE FIRST CLASS OF SOLUTIONS

The general solution of the Eq. (21) depends on the values of the parameter m , combining the effects of the thermodynamic equation of state of matter, of bulk viscosity and of the variation of the cosmological constant. Since a general solution for arbitrary m is difficult to obtain, we shall limit our study to some particular values of m , for which the general solution Eq. (21) can be expressed in an exact analytical form.

As a first class of solutions we consider the solutions generated by the choice $m = 0$, corresponding to $s = (1 + \beta)/2$, with $0 < \beta < 1$ and $1/2 < s < 1$. In this case the evolution of the bulk viscous pressure is directly coupled to the time variation of the gravitational constant, which is described by the parameter β .

Then the general solution of Eq. (21) can be obtained immediately in the form:

$$y = c_+ e^{m+\eta} + c_- e^{m-\eta}, \quad (22)$$

where c_+ and c_- are constant of integration and we have denoted $m_{\pm} = \frac{-3-\alpha_0 \pm \sqrt{\alpha_0^2 + 18/\gamma}}{2}$.

Therefore we can express the general solution of the gravitational field equations for a FRW isotropic space-time filled with a bulk viscous cosmological fluid in the framework of the full Israel-Stewart-Hiscock causal theory with variable cosmological and gravitational constants in the following exact parametric form:

$$t - t_0 = \int [c_+ e^{m+\eta} + c_- e^{m-\eta}]^{-2\gamma/n} d\eta, \quad (23)$$

$$H = [c_+ e^{m+\eta} + c_- e^{m-\eta}]^{2\gamma/n}, a = a_0 e^{\eta}, \quad (24)$$

$$\Lambda = 3\beta (c_+ e^{m+\eta} + c_- e^{m-\eta})^{4\gamma/n}, G = b (3\beta)^{n\beta/2} \Lambda^{-n\beta/2}, \rho = \rho_1 \Lambda^{n/2}, \quad (25)$$

$$p = \rho_1 (\gamma - 1) \Lambda^{n/2}, \xi = \alpha (\rho_1)^s \Lambda^{sn/2}, T = T_0 (\rho_1)^{\frac{\gamma-1}{\gamma}} \Lambda^{\frac{n(\gamma-1)}{2\gamma}}, \tau = \alpha (\rho_1)^{s-1} \Lambda^{\frac{n}{2}(s-1)}, \quad (26)$$

$$q = \gamma (\beta - 1) F(\eta) - 1, \quad (27)$$

$$\Pi = -\frac{(1-\beta)\gamma}{4\pi b} (c_+ e^{m+\eta} + c_- e^{m-\eta})^{2\gamma} \left[\frac{3}{2} + F(\eta) \right], \quad (28)$$

$$\left| \frac{\Pi}{p} \right| = \frac{2}{3} \frac{\gamma}{\gamma-1} \left| \frac{3}{2} + F(\eta) \right|, \quad (29)$$

$$\Sigma - \Sigma_0 = \frac{3(1-\beta)\gamma a_0^3}{k_B 4\pi b T_0 (\rho_0)^{\frac{\gamma-1}{\gamma}}} \int (c_+ e^{m+\eta} + c_- e^{m-\eta})^2 \left[\frac{3}{2} + F(\eta) \right] e^{3\eta} d\eta, \quad (30)$$

$$R_{ijkl} R^{ijkl} = 24 (c_+ e^{m+\eta} + c_- e^{m-\eta})^{8\gamma/n} \left[1 + \frac{2\gamma}{n} F(\eta) + \frac{2\gamma^2}{n^2} F^2(\eta) \right], \quad (31)$$

where t_0 , a_0 and Σ_0 are constants of integration. We have denoted $F(\eta) = \frac{c_+ m_+ e^{m+\eta} + c_- m_- e^{m-\eta}}{c_+ e^{m+\eta} + c_- e^{m-\eta}}$ and $\rho_1 = \frac{\rho_0}{(3\beta)^{n/2}}$.

IV. THE SECOND CLASS OF SOLUTIONS

Equation (21) has also other solutions describing the dynamics of the causal bulk viscous FRW universe for appropriate choice of the parameters m and γ . By means of further substitutions $v = 1/u$ and $u = \frac{dy}{d\eta}$ the second order differential equation (21) can be transformed into an Abel type differential equation

$$\frac{dv}{dy} = \left[\frac{9}{2\gamma} \left(\frac{\gamma}{2} - 1 \right) y + \frac{3\alpha_0}{2} y^{m+1} \right] v^3 + (3 + \alpha_0 y^m) v^2. \quad (32)$$

By introducing two new functions $A(y) = \frac{3+\alpha_0 y^m / (m+1)}{9(\gamma/2-1)/(2\gamma)+3\alpha_0 y^m/2}$ and $B(y) = \left[\frac{9(1-\gamma/2)y}{2\gamma} - \frac{3\alpha_0 y^{m+1}}{2} \right]^{-1}$ allows to rewrite Eq. (32) in the general form

$$\frac{dv}{dy} = -\frac{v^3}{B(y)} - \left[\frac{d}{dy} \frac{A(y)}{B(y)} \right] v^2. \quad (33)$$

By introducing a new variable

$$\sigma = \frac{1}{v} - \frac{A(y)}{B(y)}, \quad (34)$$

Eq. (33) can be rewritten in the general form

$$\frac{dy}{d\sigma} = \sigma B(y) + A(y), \quad (35)$$

or equivalently

$$\frac{dy}{d\sigma} = \frac{\sigma}{[9(1-\gamma/2)y/(2\gamma)-3\alpha_0 y^{m+1}/2]} + \frac{3+\alpha_0 y^m / (m+1)}{9(\gamma/2-1)/(2\gamma)+3\alpha_0 y^m/2}. \quad (36)$$

Since the energy density of the early universe is supposed to be extremely high, we can consider the extreme limit of a stiff cosmological fluid with equilibrium pressure equal to the energy density and then the parameter $\gamma = 2$. In the case $m = -2$, that is $3s = 1 + \beta$, $0 < \beta < 1$ and $1/3 < s < 2/3$, Eq. (36) becomes a Riccati type differential equation:

$$\frac{dy}{d\sigma} = \frac{2}{\alpha_0} y^2 - \frac{2}{3\alpha_0} y\sigma - \frac{2}{3}. \quad (37)$$

A particular solution of Eq. (37) is given by

$$y = -\sigma \left(\sigma^2 / \alpha_0 + 3/2 \right)^{-1}. \quad (38)$$

In the following we denote $\Delta(\sigma) = (\sigma^2 / \alpha_0 + 3/2)^{-1}$. Therefore the general solution of Eq. (37) is

$$y = -\sigma \Delta(\sigma) + \frac{\Delta^2(\sigma) e^{-\frac{\sigma^2}{3\alpha_0}}}{C_1 - \frac{2}{\alpha_0} \int \Delta^2(\sigma) e^{-\frac{\sigma^2}{3\alpha_0}} d\sigma}, \quad (39)$$

where C_1 is an arbitrary constant of integration.

The general solution of the gravitational field equations for a flat FRW space-time filled with a bulk viscous cosmological fluid and variable G and Λ can be obtained in the following exact parametric form, with σ taken as parameter:

$$t(\sigma) - t_0 = -\frac{2}{3\alpha_0} \int y^{2\beta-1}(\sigma) d\sigma, \quad (40)$$

$$H = y^{2(1-\beta)}(\sigma), a = a_0 \exp \left[-\frac{2}{3\alpha_0} \int y(\sigma) d\sigma \right], \quad (41)$$

$$\Lambda = 3\beta y^{4(1-\beta)}(\sigma), G = b y^{-4\beta}(\sigma), \rho = \rho_0 y^4(\sigma), \quad (42)$$

$$p = \rho_0 y^4(\sigma), \xi = \alpha(\rho_0)^s y^{4s}(\sigma), T = T_0(\rho_0)^{1/2} y^2(\sigma), \tau = \alpha(\rho_0)^{s-1} y^{4(s-1)}(\sigma), \quad (43)$$

$$q = 2(1 - \beta) \left[3 - \frac{\sigma}{y(\sigma)} - \frac{\alpha_0}{y^2(\sigma)} \right] - 1, \quad \Pi = \frac{1 - \beta}{4\pi b} y^2(\sigma) [3y^2(\sigma) - 2\sigma y(\sigma) - 2\alpha_0], \quad (44)$$

$$\left| \frac{\Pi}{p} \right| = \frac{2}{3} \left| \frac{3y^2(\sigma) - 2\sigma y(\sigma) - 2\alpha_0}{y^2(\sigma)} \right|, \quad (45)$$

$$\Sigma - \Sigma_0 = \frac{(1 - \beta) a_0^3}{2\pi b k_B \alpha_0 T_0 (\rho_0)^{1/2}} \int y(\sigma) [3y^2(\sigma) - 2\sigma y(\sigma) - 2\alpha_0] \exp \left[-\frac{2}{\alpha_0} \int y(\sigma) d\sigma \right] d\sigma, \quad (46)$$

$$R_{ijkl} R^{ijkl} = 24y^{4-8\beta} \left[y^4 + 2(\beta - 1)y^2(3y^2 - y\sigma - \alpha_0) + 2(\beta - 1)^2(3y^2 - y\sigma - \alpha_0)^2 \right], \quad (47)$$

where t_0 , a_0 and Σ_0 are constants of integration.

V. DISCUSSIONS AND FINAL REMARKS

In the present paper we have considered the evolution of a full causal bulk viscous Universe in the presence of variable gravitational and cosmological "constants". We have adopted a particular model for the time variation of the cosmological constant, assuming that it is a quadratic function of the Hubble parameter H . Consequently, the gravitational constant has also a power dependence on the Hubble parameter. In both classes of solutions the evolution of bulk viscosity coefficient is related to that of G and Λ . While the first solution describes a cosmological fluid with barotropic equation of state for arbitrary γ , the range of application of the second solution is restricted to a very high energy density fluid, obeying the stiff equation of state.

The causal bulk viscous Universe described by the first class of solutions starts its evolution from a singular state with zero value of the scale factor a , $a(t_0) = 0$, and infinite value of the energy density, $\rho(t_0) \rightarrow \infty$.

The existence of a singular initial state is also showed by the behavior of the invariant $R_{ijkl} R^{ijkl}$, represented in Fig.1, which is singular function of time.

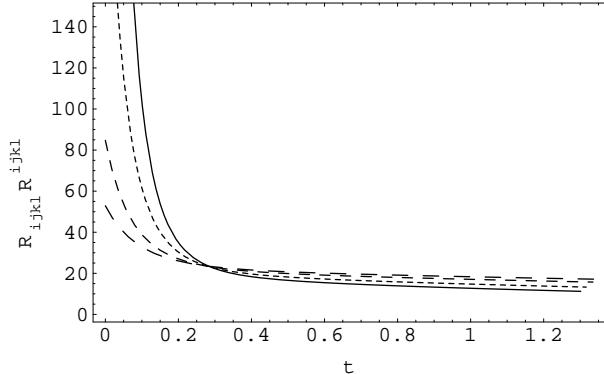


FIG. 1. Time variation of the invariant $R_{ijkl} R^{ijkl}$ for the first class of solutions for a radiation filled bulk viscous Universe ($\gamma = 4/3$), for different values of the parameter β : $\beta = 0.1$ (solid curve), $\beta = 0.3$ (dotted curve), $\beta = 0.5$ (dashed curve), $\beta = 0.6$ (long dashed curve). We have chosen the integration constants so that $c_+ = c_- = 1$ and assumed that $\alpha_0 = 1$.

The evolution of the Universe is expansionary, with the scale factor an increasing function of time. The Hubble parameter, presented in Fig. 2, is a monotonically decreasing function of time, similar to the energy density of the cosmological fluid, represented in Fig. 3.

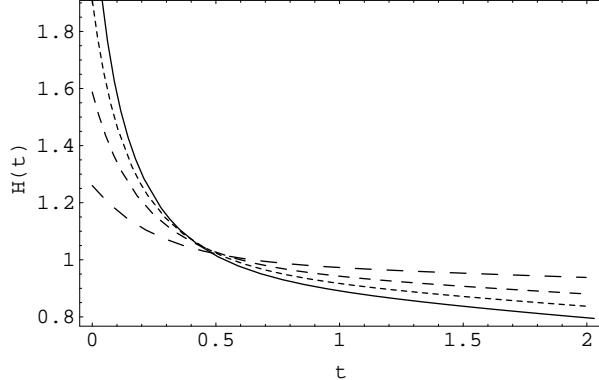


FIG. 2. Evolution of the Hubble parameter H for the first class of solutions for a radiation filled bulk viscous Universe ($\gamma = 4/3$), for different values of the parameter β : $\beta = 0.1$ (solid curve), $\beta = 0.3$ (dotted curve), $\beta = 0.5$ (dashed curve), $\beta = 0.6$ (long dashed curve). We have chosen the integration constants so that $c_+ = c_- = 1$ and assumed that $\alpha_0 = 1$.

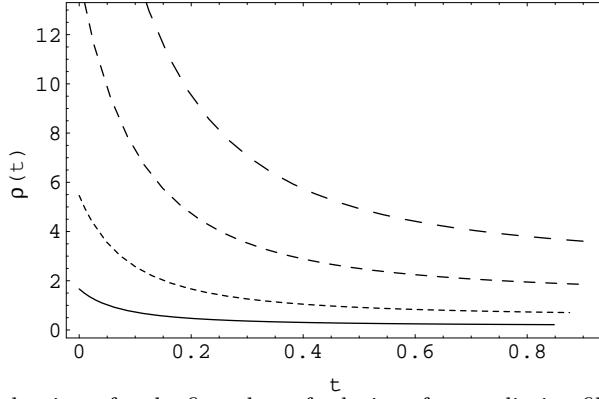


FIG. 3. Dynamics of the energy density ρ for the first class of solutions for a radiation filled bulk viscous Universe ($\gamma = 4/3$), for different values of the parameter β : $\beta = 0.1$ (solid curve), $\beta = 0.3$ (dotted curve), $\beta = 0.5$ (dashed curve), $\beta = 0.6$ (long dashed curve). We have chosen the integration constants so that $c_+ = c_- = 1$ and assumed that $\alpha_0 = 1$.

The bulk viscous pressure Π , shown in Fig. 4, satisfies the condition $\Pi < 0$ only for time intervals greater than an initial value t_1 .

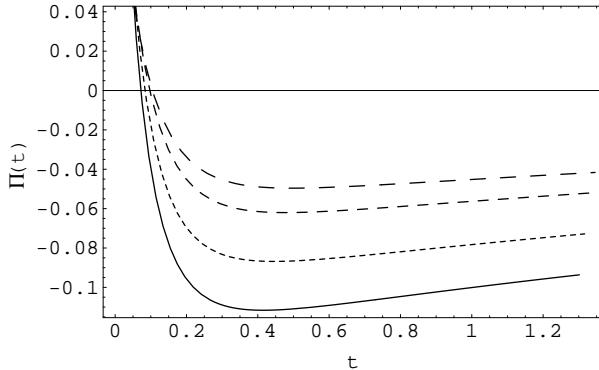


FIG. 4. Variation of the bulk viscous pressure Π of the cosmological fluid for the first class of solutions for a radiation filled bulk viscous Universe ($\gamma = 4/3$), for different values of the parameter β : $\beta = 0.1$ (solid curve), $\beta = 0.3$ (dotted curve), $\beta = 0.5$ (dashed curve), $\beta = 0.6$ (long dashed curve). We have chosen the integration constants so that $c_+ = c_- = 1$ and assumed that $\alpha_0 = 1$.

Hence this model can describe the dynamics of the causal bulk viscous Universe with variable gravitational and cosmological constants only for a finite time interval.

The deceleration parameter q , represented in Fig. 5, shows an initial non-inflationary evolution, with $q > 0$, but in the large time limit the Universe ends in an inflationary epoch.

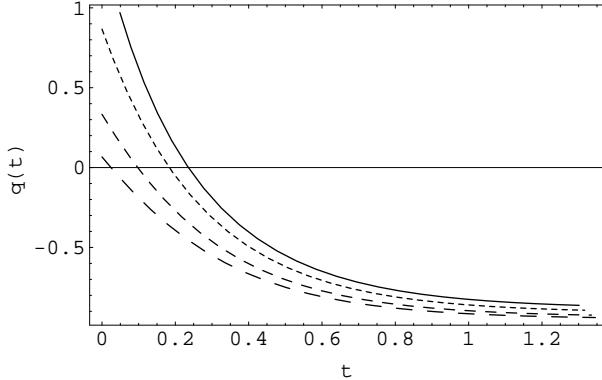


FIG. 5. Dynamics of the deceleration parameter q for the first class of solutions for a radiation filled bulk viscous Universe ($\gamma = 4/3$), for different values of the parameter β : $\beta = 0.1$ (solid curve), $\beta = 0.3$ (dotted curve), $\beta = 0.5$ (dashed curve), $\beta = 0.6$ (long dashed curve). We have chosen the integration constants so that $c_+ = c_- = 1$ and assumed that $\alpha_0 = 1$.

During the viscous effects dominated era a large amount of comoving entropy is produced. Fig. 6 shows the time variation of the entropy.

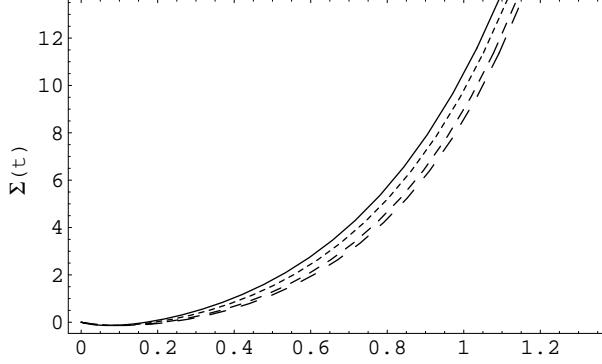


FIG. 6. Time variation of the comoving entropy Σ for the first class of solutions for a radiation filled bulk viscous Universe ($\gamma = 4/3$), for different values of the parameter β : $\beta = 0.1$ (solid curve), $\beta = 0.3$ (dotted curve), $\beta = 0.5$ (dashed curve), $\beta = 0.6$ (long dashed curve). We have chosen the integration constants so that $c_+ = c_- = 1$ and assumed that $\alpha_0 = 1$.

The time variation of the cosmological and gravitational constants is represented in Figs. 7 and 8.

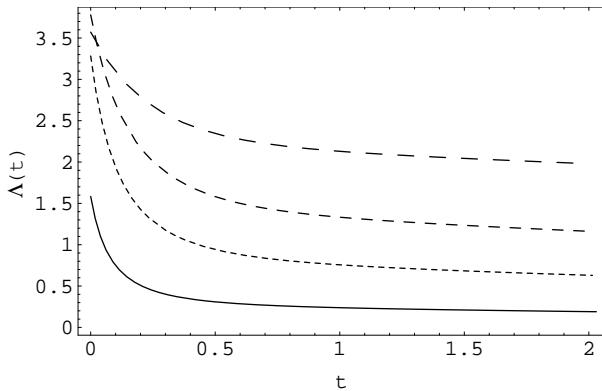


FIG. 7. Time variation of the cosmological constant $\Lambda(t)$ for the first class of solutions for a radiation filled bulk viscous Universe ($\gamma = 4/3$), for different values of the parameter β : $\beta = 0.1$ (solid curve), $\beta = 0.3$ (dotted curve), $\beta = 0.5$ (dashed curve), $\beta = 0.6$ (long dashed curve). We have chosen the integration constants so that $c_+ = c_- = 1$ and assumed that $\alpha_0 = 1$.

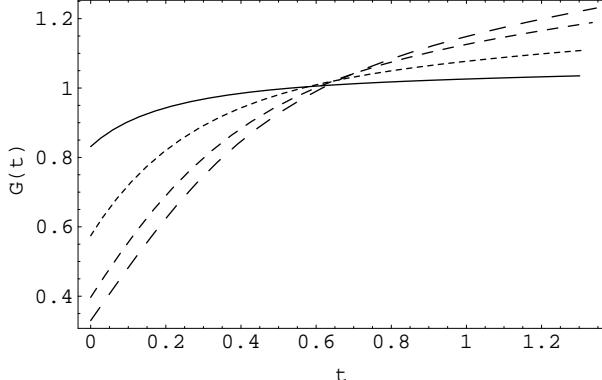


FIG. 8. Time variation of the gravitational constant $G(t)$ for the first class of solutions for a radiation filled bulk viscous Universe ($\gamma = 4/3$), for different values of the parameter β : $\beta = 0.1$ (solid curve), $\beta = 0.3$ (dotted curve), $\beta = 0.5$ (dashed curve), $\beta = 0.6$ (long dashed curve). We have chosen the integration constants so that $c_+ = c_- = 1$ and assumed that $\alpha_0 = 1$.

The cosmological constant is a decreasing function of time, while the gravitational constant G tends in the large time to a constant value. But in this model the ratio of the bulk viscous pressure and of the thermodynamical pressure is greater than one during the inflationary period. Consequently during this period the model is not consistent thermodynamically.

The evolution of the stiff cosmological fluid filled bulk viscous Universe, described by the second class of solutions, also starts from a singular state, with $a(t_0) = 0$. The invariant $R_{ijkl}R^{ijkl}$, represented in Fig. 9, is also singular in the large time limit.

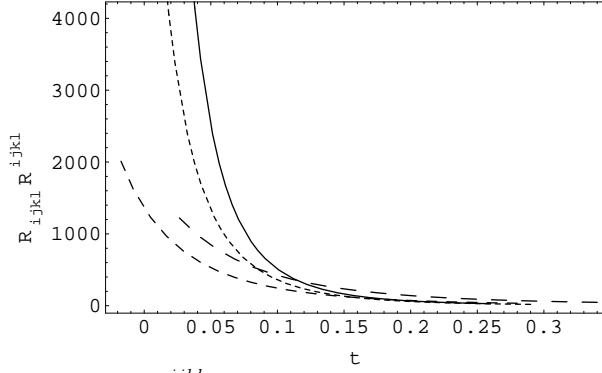


FIG. 9. Time variation of the invariant $R_{ijkl}R^{ijkl}$ for the second class of solutions describing a stiff fluid filled bulk viscous Universe ($\gamma = 2$), for different values of the parameter β : $\beta = 0.1$ (solid curve), $\beta = 0.3$ (dotted curve), $\beta = 0.5$ (dashed curve), $\beta = 0.6$ (long dashed curve). We assumed that $\alpha_0 = 1$ and we have chosen the arbitrary integration constant $C_1 = 1/2$.

The evolution of the Universe is expansionary, with the scale factor, represented in Fig. 10, an increasing function of time.

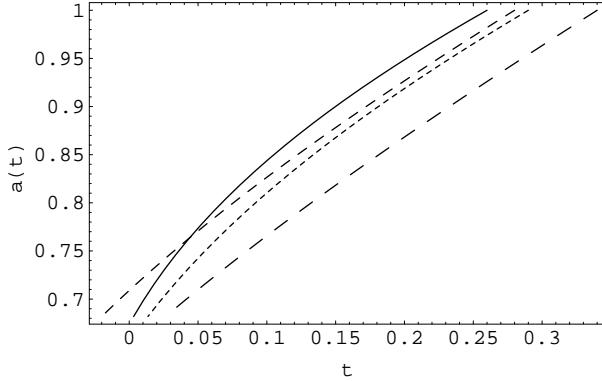


FIG. 10. Time evolution of the scale factor a for the second class class of solutions describing a stiff fluid filled bulk viscous Universe ($\gamma = 2$), for different values of the parameter β : $\beta = 0.1$ (solid curve), $\beta = 0.3$ (dotted curve), $\beta = 0.5$ (dashed curve), $\beta = 0.6$ (long dashed curve). We assumed that $\alpha_0 = 1$ and we have chosen the arbitrary integration constant $C_1 = 1/2$.

The Hubble parameter and the energy density, represented in Fig. 11, are monotonically decreasing functions of time.

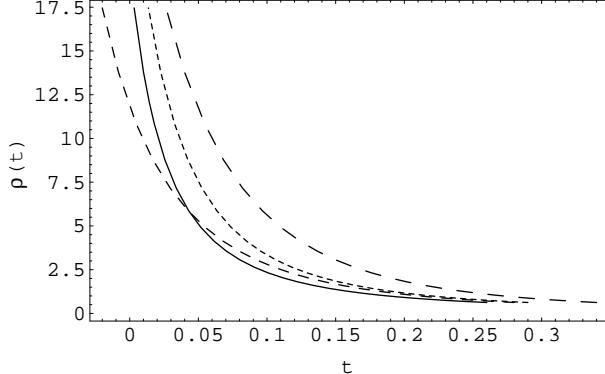


FIG. 11. Dynamics of the energy density ρ for the second class class of solutions describing a stiff fluid filled bulk viscous Universe ($\gamma = 2$), for different values of the parameter β : $\beta = 0.1$ (solid curve), $\beta = 0.3$ (dotted curve), $\beta = 0.5$ (dashed curve), $\beta = 0.6$ (long dashed curve). We assumed that $\alpha_0 = 1$ and we have chosen the arbitrary integration constant $C_1 = 1/2$.

The bulk viscous pressure Π satisfies the condition $\Pi < 0$ for all time intervals and for all values of the parameter, as can be seen from Fig. 12.

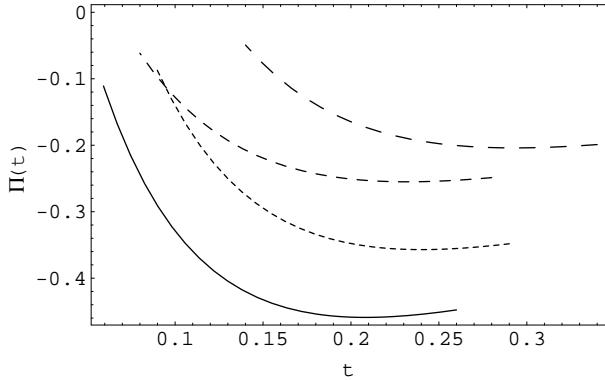


FIG. 12. Time variation of the bulk viscous pressure Π for the second class class of solutions describing a stiff fluid filled bulk viscous Universe ($\gamma = 2$), for different values of the parameter β : $\beta = 0.1$ (solid curve), $\beta = 0.3$ (dotted curve), $\beta = 0.5$ (dashed curve), $\beta = 0.6$ (long dashed curve). We assumed that $\alpha_0 = 1$ and we have chosen the arbitrary integration constant $C_1 = 1/2$.

The deceleration parameter q , represented in Fig. 13, satisfies in this model the condition $q > 0$ for all times and in the acceptable range of the physical parameters.

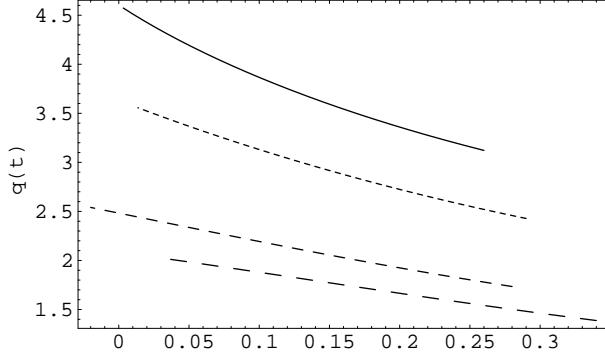


FIG. 13. Time dependence of the deceleration parameter q in the second class class of solutions describing a stiff fluid filled bulk viscous Universe ($\gamma = 2$), for different values of the parameter β : $\beta = 0.1$ (solid curve), $\beta = 0.3$ (dotted curve), $\beta = 0.5$ (dashed curve), $\beta = 0.6$ (long dashed curve). We assumed that $\alpha_0 = 1$ and we have chosen the arbitrary integration constant $C_1 = 1/2$.

Therefore the evolution of the high density causal bulk viscous Universe is non-inflationary. Consequently, the model is consistent thermodynamically satisfying the condition of the smallness of the bulk viscous pressure. The time evolutions of the cosmological constant represented in Figs. 14 is similar to the behavior of this quantity in the first model, with the cosmological constant a decreasing function of time, while in the large time limit the gravitational constant, showed in Fig. 15, is a slowly increasing function of time.

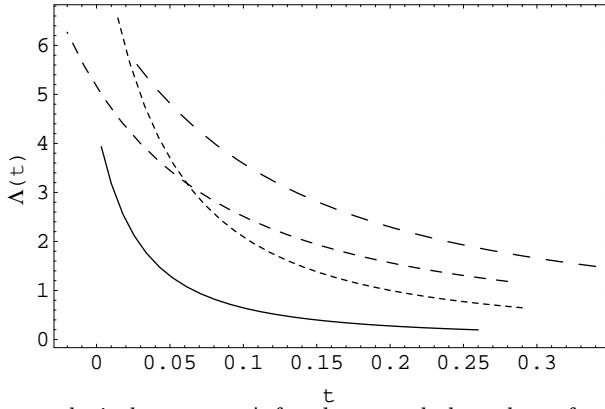


FIG. 14. Time variation of the cosmological constant Λ for the second class class of solutions describing a stiff fluid filled bulk viscous Universe ($\gamma = 2$), for different values of the parameter β : $\beta = 0.1$ (solid curve), $\beta = 0.3$ (dotted curve), $\beta = 0.5$ (dashed curve), $\beta = 0.6$ (long dashed curve). We assumed that $\alpha_0 = 1$ and we have chosen the arbitrary integration constant $C_1 = 1/2$.

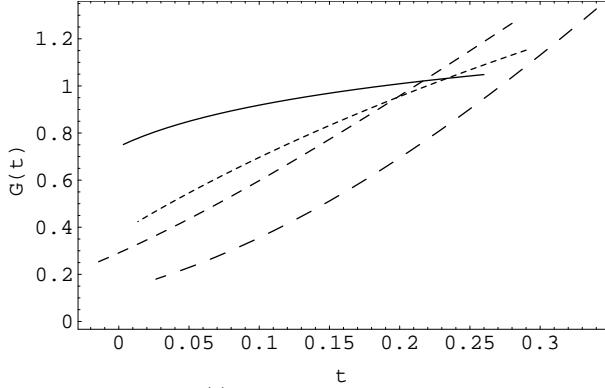


FIG. 15. Evolution of the gravitational constant $G(t)$ for the second class class of solutions describing a stiff fluid filled bulk viscous Universe ($\gamma = 2$), for different values of the parameter β : $\beta = 0.1$ (solid curve), $\beta = 0.3$ (dotted curve), $\beta = 0.5$ (dashed curve), $\beta = 0.6$ (long dashed curve). We assumed that $\alpha_0 = 1$ and we have chosen the arbitrary integration constant $C_1 = 1/2$.

The possibility that the cosmological constant and the gravitational coupling are not real constants is an intriguing possibility, which has intensively been investigated in the physical literature. It is a very plausible hypothesis that these effects were much stronger in the early Universe, when dissipative effects also played an important role in the dynamics of the cosmological fluid. Hence the solutions obtained in the present paper could give an appropriate description of the early period of our Universe.

- [1] A. G. Riess et al, *Astron. J.* **116** (1998) 1099.
- [2] S. Perlmutter et al, *Astrophys. J.* **517** (1999) 517.
- [3] S. Perlmutter et al, *Nature* **391** (1998) 51.
- [4] J. P. Ostriker and P. J. Steinhardt, *Nature* **377** (1995) 600.
- [5] S. Weinberg, *Rev. Mod. Phys.* **61** (1989) 1; S. Weinberg, astro-ph/0005265; M. Gasperini, *Phys. Lett.* **B194** (1987) 347; M. Gasperini, *Class. Quantum Grav.* **5** (1988) 521.
- [6] P. A. M. Dirac, *Proc. R. Soc. London* **A165** (1938) 199.
- [7] A-M. M. Abdel-Rahman, *Nuovo Cimento* **B102** (1988) 225.
- [8] A-M. M. Abdel-Rahman, *Phys. Rev.* **D45** (1992) 3497.
- [9] J. D. Barrow, *Phys. Lett.* **B180** (1986) 335.
- [10] J. D. Barrow and P. Parsons, *Phys. Rev.* **D55** (1997) 1906.
- [11] A. Beesham, *Gen. Rel. Grav.* **26** (1993) 159.
- [12] M. S. Berman, *Phys. Rev.* **D43** (1991) 1075.
- [13] J. C. Carvalho, J. A. S. Lima and I. Waga, *Phys. Rev.* **D46** (1992) 2404.
- [14] W. Chen and Y. S. Wu, *Phys. Rev.* **D41** (1990) 695.
- [15] T. Damour, G. W. Gibbons and J. H. Taylor, *Phys. Rev. Lett.* **61** (1988) 1151.
- [16] T. B. Enginol, *Phys. Lett.* **A139** (1989) 127.
- [17] D. Kalligas, P. S. Wesson and C. W. F. Everitt, *Gen. Rel. Grav.* **24** (1992) 351.
- [18] D. Kalligas, P. S. Wesson and C. W. F. Everitt, *Gen. Rel. Grav.* **27** (1995) 645.
- [19] Y. K. Lau, *Aust. J. Phys.* **38** (1985) 547.
- [20] Y. K. Lau and S. J. Prokhorov, *Aust. J. Phys.* **39** (1986) 339.
- [21] J. A. S. Lima and J. M. F. Maia, *Phys. Rev.* **D49** (1994) 5597.
- [22] J. A. S. Lima and M. Trodden, *Phys. Rev.* **D53** (1996) 4280.
- [23] P. J. E. Peebles and B. Ratra, *Astrophys. J.* **325** (1988) L17.
- [24] T. Singh, A. Beesham and W. S. Mbokazi, *Gen. Rel. Grav.* **30** (1998) 573.
- [25] R. F. Sistero, *Gen. Rel. Grav.* **23** (1991) 1265.
- [26] J. A. S. Lima and J. M. F. Maia, *Phys. Rev.* **D49** (1994) 5597.
- [27] I. Waga, *Astrophys. J.* **414** (1993) 436.
- [28] C. Eckart, *Phys. Rev.* **58** (1940) 919.
- [29] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, (1987) Butterworth Heinemann.
- [30] W. Israel, *Ann. Phys.* **100** (1976) 310.
- [31] W. Israel and J. M. Stewart, *Phys. Lett.* **A58** (1976) 213.
- [32] W. A. Hiscock and L. Lindblom, *Ann. Phys.* **151** (1989) 466.
- [33] W. A. Hiscock and J. Salmonson, *Phys. Rev.* **D43** (1991) 3249.
- [34] R. Maartens, *Class. Quantum Grav.* **12** (1995) 1455.
- [35] R. Maartens, *Causal thermodynamics in relativity*, astro-ph/9609119.
- [36] L. P. Chimento and A. S. Jakubi, *Class. Quantum Grav.* **14** (1997) 1811.
- [37] L. P. Chimento and A. S. Jakubi, *Int. J. Mod. Phys.* **D7** (1998) 177.
- [38] M. K. Mak and T. Harko, *Gen. Rel. Grav.* **30** (1998) 1171; M. K. Mak and T. Harko, *Gen. Rel. Grav.* **31** (1999) 273.
- [39] M. K. Mak and T. Harko, *J. Math. Phys.* **39** (1998) 5458.
- [40] T. Harko and M. K. Mak, *Int. J. Theor. Phys.* **38** (1999) 1561.
- [41] M. K. Mak and T. Harko, *Aust. J. Phys.* **52** (1999) 659.
- [42] M. K. Mak and T. Harko, *Int. J. Mod. Phys.* **D9** (2000) 97.
- [43] M. K. Mak and T. Harko, *Aust. J. Phys.* **53** (2000) 241.
- [44] M. K. Mak and T. Harko, *Int. J. Mod. Phys.* **D9** (2000) 475.
- [45] R. Maartens and J. Triginer, *Phys. Rev.* **D56** (1997) 4640.
- [46] A. A. Coley and R. J. van den Hoogen, *Class. Quantum Grav.* **12** (1995) 1977.
- [47] A. A. Coley and R. J. van den Hoogen, *Phys. Rev.* **D54** (1996) 1393.
- [48] A. Di Prisco, L. Herrera and J. Ibanez, *Phys. Rev.* **D63** (2001) 023501.

- [49] J. M. Overduin and F. I. Cooperstock, *Phys. Rev.* **D58** (1998) 043506.
- [50] Abdussatar and R. G. Vishwakarma, *Class. Quantum Grav.* **14** (1997) 945.
- [51] A. I. Arbab, *Gen. Rel. Grav.* **29** (1997) 61.
- [52] A. Beesham, *Phys. Rev.* **D48** (1993) 3539.
- [53] V. Mendez and D. Pavon, *Gen. Rel. Grav.* **28** (1996) 679.
- [54] A. I. Arbab and A. Beesham, *Gen. Rel. Grav.* **32** (2000) 615.
- [55] A. I. Arbab and A.-M. M. Abdel-Rahman, *Phys. Rev.* **D50** (1994) 7725.
- [56] J. A. S. Lima and J. C. Carvalho, *Gen. Rel. Grav.* **26** (1994) 909.
- [57] J. M. Salim and I. Waga, *Class. Quantum Grav.* **10** (1993) 1767.